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Conformations and persistence lengths of vortex filaments in homogeneous turbulence

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Abstract. We propose a theory describing the conformations of the coherent vortex filaments observed in homogeneous and isotropic turbulence. These objects are pictured as a gas of non-interacting singular structures enveloped in a given background flow characterized by a self-similar energy spectrum. In a general way, we show that filament conformation can be mapped to a random walk problem with long-range correlations. Its Flory exponent is related to a correlation exponent within a self-consistent approximation, without invoking thermal equilibrium arguments. The filament fractal dimension and its energy spectrum also obey a simple relation. The filaments are locally linear and, at scales smaller than a persistence length, form rather straight lines. Under the assumption that these defects are special, intense realizations of the vorticity background statistics, we evaluate persistence lengths that show good agreement with previous simulation results at intermediate Reynolds numbers.

1. Introduction

The most intense vorticity regions encountered in turbulent flows at high Reynolds numbers are likely to have the shape of thin filaments [1–8]. Some structures, observed both in experiments and numerical simulations, have a relatively long lifetime compared with a typical eddy turnover time and are thus considered as coherent. Numerical studies show that these filaments do not contribute predominantly to the motion of the total flow, as they contain a small part of the total kinetic energy and enstrophy [3]. They suggest that they are simply intense events of the background flow. Nevertheless, these vortices probably have close connections with the intermittent statistics in flows that cannot be explained by the Kolmogorov [9] theory. It remains a difficult task to explain their fine spatial structure. However, the understanding of some properties of turbulent flows has been improved by the study of much simpler one-vortex instability problems. For instance, it has been shown that a Burger's vortex submitted to a strain rate can reproduce the intermittent exponents of homogeneous isotropic turbulence [10]. The intermittency model of She and Leveque [11] also describes the whole set of structure functions exponents in terms of vortex filaments.

According to various simulation results (see e.g. [6]), intense turbulent vortices are usually pictured as linear objects. The roughly cylindrical vortices observed in numerical simulations have been interpreted to be possibly the asymptotic form of initial spiral vortex sheets submitted

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to a strain. Contrary to spiral sheets, filaments have a small core, whose thickness is of the order of a few dissipation scales [3, 5]. Therefore, it seems that their most relevant spectral properties are rather contained in their longitudinal structure than in their transverse cross section. Indeed, direct observation shows that the shape of the filaments is more complex than a straight line, and may even have a small radius of curvature. It is noticeable at large Reynolds numbers, when destabilized vortex tubes take a complex shape before bursting.

We adopt a scheme where coherent vortices are represented as a gas of non-interacting structures enveloped in a relatively simple background flow. The background is described, say, by an energy spectrum of the Kolmogorov kind, and we will assume that the vortices surrounded by such a disordered flow should keep some of their statistical properties. Obviously, filaments introduce intermittency effects that differ from simple Gaussian statistics, and that become important when looking at the high-order moments of velocity differences. Conversely, She *et al* [1] proved that coherent structures cannot develop in a Gaussian random velocity field. However, since second-order structure functions are close to Kolmogorov predictions in experiments and in most of the intermittency models, we will suppose that a single filament reproduces a vortical structure implicitly contained in the background, provided that the moments under consideration are of low order. Similar hypotheses have been used successfully in [10] for random strained vortices. The idea that coherent vortices are not necessary, in contradiction to Kolmogorov predictions, is not new. It is well known that spiral vortices solutions of the Navier–Stokes equation can organize according to a Kolmogorov energy spectrum [12]. We wish to apply scaling considerations to study the spatial conformations of non-straight thin vortices. Some concepts borrowed from polymer physics are thus useful. Polymers and random walks are examples of fractal objects and, for this property, have been formally compared with the scale-invariant spectra of turbulence [13]. We will focus here on the root mean square of the distance that separate two points on the filament line, a moment of order two extensively investigated in the context of polymers [14].

As shown in the next section, the study of filament conformations can be mapped to a random walk problem with long-range correlations. For very large Reynolds number, we find that an isolated filament has a fractal structure. Within a self-consistent approximation, we derive its Flory exponent as a function of a tangent vector correlation function exponent. The kinetic energy spectrum generated by the filaments is then determined in section 3. We further assume that the vorticity two-point autocorrelation function on the filament is the same as in the background flow. In section 4, we introduce a persistence length and show that filaments form fairly straight lines at intermediate Reynolds numbers ($\sim 10^2$), in agreement with previous numerical studies [4, 5]. Our conclusions are presented in section 5.

2. Correlation exponent and scaling properties

Let us consider, submerged in a turbulent background flow, a gas of statistically independent vortex filaments modelling intense events. We assume that the volume fraction occupied by these vortices is sufficiently small so that they contribute negligibly to the energy spectrum of the whole flow, mainly given by the background. The background flow is characterized by two very different length scales: an energy injection scale l and a dissipative scale $a \ll l$. Simulation results show that the cross section of vortex filaments is of the order of a few dissipation lengths a, while their total length is comparable to l [3]. Each filament can be schematically described by a space curve $\boldsymbol{r} = \boldsymbol{r}_0(s)$, s being a curvilinear coordinate. We define the two-point autocorrelation function of any vorticity distribution as

$$R_\omega(r) = \frac{\langle \vec{\omega}(\boldsymbol{x}) \cdot \vec{\omega}(\boldsymbol{x}+\boldsymbol{r}) \rangle}{\langle \vec{\omega}^2(\boldsymbol{x}) \rangle} \tag{2.1}$$

which is a function of $r = |\boldsymbol{r}|$ only for an isotropic, homogeneous flow. By considering the vorticity distributed along the filament subset only, definition (2.1) turns into

$$R_\omega^{(f)}(r) = \langle \boldsymbol{t}(s) \cdot \boldsymbol{t}(s') \rangle \tag{2.2}$$

where the index (f) refers to the filaments, and $\boldsymbol{t}(s) = \mathrm{d}\boldsymbol{r}_0(s)/\mathrm{d}s$ is the local unit tangent vector of the line curve $\boldsymbol{r}_0$. The two points considered on the right-hand side of equation (2.2) are such that $|\boldsymbol{r}_0(s) - \boldsymbol{r}_0(s')| = r$, and belong to the same, isolated filament.

If the filaments are long enough, it is natural to introduce a Flory exponent ν to characterize their large-scale conformations. (The structure of the filaments at intermediate scales will be further analyzed in section 4.) ν is defined by

$$\langle R^2 \rangle \sim L^{2\nu} \tag{2.3}$$

where $\langle R^2 \rangle^{1/2}$ is the rms distance that separates two points located on the line at coordinates s and s', such that $|s - s'| = L$. The value $\nu = 1$ corresponds to a straight, rod-like line; $\nu = 1/2$ is the exponent of ideal polymer chains or Gaussian random walks [14].

Since a coherent vortex filament is responsible for additional order in the vortex distribution, we assume that correlations between tangent vectors decay slowly with separation distance, according to the general form

$$\langle \boldsymbol{t}(\boldsymbol{x}) \cdot \boldsymbol{t}(\boldsymbol{x}') \rangle \sim |\boldsymbol{x} - \boldsymbol{x}'|^{-\alpha} \tag{2.4}$$

where $\boldsymbol{x}$ and $\boldsymbol{x}'$ are two points of $\boldsymbol{r}_0$ and α is a given positive correlation exponent. Relation (2.4) is assumed to hold, say, for $a \ll |\boldsymbol{x} - \boldsymbol{x}'| \ll l$. Further justification will be given for such a form. We are left with a random walk problem that consists in finding the Flory exponent of a line that satisfies the relation (2.4).

Diffusion processes that involve long-range correlations, as above, generally result in anomalous (or non-Brownian) diffusion exponents, such that $\nu > 1/2$ [15]. Let us note that the end-to-end distance (2.3) is

$$\langle R^2 \rangle = \int_0^L \mathrm{d}s \int_0^L \mathrm{d}s' \, \langle \boldsymbol{t}(s) \cdot \boldsymbol{t}(s') \rangle. \tag{2.5}$$

The difficulty to overcome is the fact that we do not have a full equation for $\boldsymbol{t}(s)$. Hence, no rigorous solution can be found for ν. It differs from classical equilibrium problems, such as Brownian diffusion or polymers, where Langevin equations with thermal noise or an ensemble Gibbs measure can be invoked. However, with the help of the only average relation (2.4), the problem can still be solved within a self-consistent approximation. It consists in invoking the argument that two points $\boldsymbol{x}$ and $\boldsymbol{x}'$ located on the line are, on average, separated by a line segment of length $|s - s'| \sim |\boldsymbol{x} - \boldsymbol{x}'|^{1/\nu}$. It yields the self-consistent relation

$$\langle \boldsymbol{t}(s) \cdot \boldsymbol{t}(s') \rangle \sim |s - s'|^{-\nu\alpha}. \tag{2.6}$$

Combining equations (2.5) and (2.6), one finds, after identification with definition (2.3):

$$\nu = \begin{cases} 2/(2+\alpha) & \text{if} \quad 0 \leqslant \alpha \leqslant 2 \\ 1/2 & \text{if} \quad \alpha \geqslant 2. \end{cases} \tag{2.7}$$

As usual in anomalous diffusion problems, the Flory exponent $\nu = 1/2$ corresponding to the uncorrelated case is recovered as soon as correlations (2.4) decrease faster than a limiting law (here r^{-2})†. As expected, $\alpha = 0$ corresponds to rod-like filaments. Hence, correlations given by the law (2.4) accounts for all filament conformations between a straight line and a Gaussian random walk.

† Here, the exponent cannot be lower than 1/2, the correlation function being positive. However, $\nu < 1/2$ may occur in other random walk problems if an effective attraction is present.

We note in the appendix that self-consistent approximations in random walk problems rely on quite general scaling properties. Exponents with analytical expressions identical to the result of equation (2.7) appear in other contexts, such as in problems of diffusion in random media [15], see the appendix.

We now turn to the turbulent background flow of kinetic energy spectrum $E^{(b)}(q)$. In the incompressible and three-dimensional case, the vorticity correlation function in Fourier space and the energy spectrum are simply proportional, i.e. $\langle\vec{\omega}(\boldsymbol{q})\cdot\vec{\omega}(-\boldsymbol{q})\rangle \propto E^{(b)}(q)$ [16]. The background vorticity autocorrelation function is, from (2.1),

$$R_\omega^{(b)}(r) = \left(\int_0^\infty \mathrm{d}q\, q^2 E^{(b)}(q)\frac{\sin qr}{qr}\right)\left(\int_0^\infty \mathrm{d}q\, q^2 E^{(b)}(q)\right)^{-1}. \tag{2.8}$$

The index (b) refers to the homogeneous background. We assume that the energy spectrum is self-similar between the two wavenumbers $q_l \sim 1/l$ and $q_a \sim 1/a$ ($q_l \ll q_a$). Consider the general family of spectra

$$E^{(b)}(q) \sim q^{-\mu}F(q/q_a) \qquad \text{for} \quad q > q_l \tag{2.9}$$

where $F(x)$ is a flat function for small x and decreases rapidly to zero for $x > 1$. We assume in the following that $\mu < 3$ (which contains the Kolmogorov scaling $\mu = 5/3$). It is easy to show from equations (2.8) and (2.9) that $R_\omega^{(b)}$ satisfies the scaling law (see e.g. [16])

$$R_\omega^{(b)}(r) = \left(\frac{r}{a}\right)^{-(3-\mu)} f(r/l) \qquad r \gg a \tag{2.10}$$

where f is a scaling function that essentially depends on the (non-universal) behaviour of $E^{(b)}(q)$ for $q < q_l$. According to the phenomenological theory of Kolmogorov, the ratio l/a diverges at large Reynolds numbers [16]. Therefore, in this limit, which will be considered in the following, the correlation function $R_\omega^{(b)}$ simply reduces to an inverse power law of the variable r/a, i.e. to a form (2.4) with $\alpha = 3-\mu$. This is an *a posteriori* check for the relevance of the general form (2.4) for vortex correlations.

As discussed in the introduction, intermittency models or experiments usually predict that second-order moments are close to the Kolmogorov scaling. In section 4, we will assume self-consistently that the two-point vorticity correlation function on an isolated filament and in the background flow are the same, i.e. $R_\omega^{(f)} = R_\omega^{(b)}$. In that case, $\alpha = 3 - \mu$.

3. Energy spectrum of the filaments

We now wish to find out the energy spectrum generated by a single filament, knowing its Flory exponent calculated previously. This generalizes the classical k^{-1} spectrum corresponding to straight vortices. Yet, let us recall that, if the volume fraction occupied by the filaments is small, they contribute negligibly to the total background spectrum. We follow a method inspired by [17], where a similar analysis was developed, although in a different context.

In equation (2.4) the coordinates are restricted to be on a fractal support of dimension lower than 3 and therefore relation (2.8) between the spectrum and the correlations cannot be valid for $R_\omega^{(f)}$. It can be useful to replace the filamentary vorticity field $\omega^{(f)}$ by an effective field ω^* with a support of dimension 3, chosen such that

$$\int_{|\boldsymbol{x}|<r} \mathrm{d}\boldsymbol{x}\,\langle\omega^{(f)}(\mathbf{0})\cdot\omega^{(f)}(\boldsymbol{x})\rangle = \int_{|\boldsymbol{x}|<r} \mathrm{d}\boldsymbol{x}\,\langle\omega^{*}(\mathbf{0})\cdot\omega^{*}(\boldsymbol{x})\rangle. \tag{3.1}$$

The origin on the left-hand side is a point located on the filament. We now invoke an argument that can be found in a similar form in [17]. The volume occupied by the filament in a sphere of radius r around the origin is proportional to $r^{1/\nu}$, so that the probability a point $\boldsymbol{x}$ in the sphere belongs to the filament is proportional to $r^{1/\nu}/r^3$. With the help of equation (2.4), the left-hand side of equation (3.1) can thus be evaluated as $r^{1/\nu-3}\int_0^r \rho^2\,\mathrm{d}\rho\,\rho^{-\alpha} = r^{\delta}$, with $\delta = 1/\nu - \alpha$. Equating this with the right-hand side of equation (3.1) leads to $\langle \omega^*(\mathbf{0})\cdot\omega^*(\boldsymbol{x})\rangle \sim |\boldsymbol{x}|^{-(3-\delta)}$. If δ is positive, we deduce that the energy spectrum $E^{(f)}$ associated with the filament scales as $q^{-\delta}$—see relation (2.10). If δ is negative or zero, we simply get a white spectrum $E^{(f)}(q) \sim q^0$. Using equation (2.7), the results can be recast as

$$E^{(f)}(q) \sim q^{-\mu'} \qquad \text{with} \quad \mu' = \begin{cases} 1-\alpha/2 & \text{if} \quad 0 \leqslant \alpha \leqslant 2 \\ 0 & \text{if} \quad \alpha \geqslant 2. \end{cases} \tag{3.2}$$

Note that the results can also be expressed with the Flory exponent through a single relation:

$$\mu' + \frac{1}{\nu} = 2. \tag{3.3}$$

One recovers the k^{-1} spectrum corresponding to the simple flow field produced by a straight filament ($\nu = 1$). Equation (3.3) can also be written as: $\mu' + D = 2$, where $D = 1/\nu$ represents the fractal dimension of the random walk. This last relation generalizes to $1 \leqslant D \leqslant 2$, the result derived in the particular case $D = 1$ for straight filaments by use of multifractal arguments [18].

Since a vortex line is a low-dimensional object, it cannot generate fairly ordered flows, as shown by the inequality $\mu' \leqslant 1$. One checks that the spectrum slope decreases as the Flory exponent decreases: folded filaments generate less structured flows. A result implicitly similar to equation (3.3) can be found in [17]; however, the model introduced there is different, since filaments are assumed to be the only component of the flow.

If we take the self-consistent expression $\alpha = 3 - \mu$, the Kolmogorov scaling $\mu = 5/3$ leads to a Flory exponent $\nu = 3/5$ and a spectrum $E^{(f)}(q) \sim q^{-1/3}$. We emphasize that, although the shape of the filament is determined from the statistics of the background flow, its energy spectrum is *not* that of the background. This is due to the fact that the vorticity is constrained to be distributed along a line and not uniformly throughout the volume. The value found for ν is surprisingly the same as the result given by the Flory argument for random self-avoiding walks. This similitude is coincidental, as we have not made the assumption that the filaments are self-avoiding in our model. Indeed, $\nu = 3/5$ does not means that the vortex is effectively self-avoiding, but it results from the particular correlations of the random walk. This differs from the model proposed in [17], where self-avoidance on a lattice is explicitly assumed, together with a Gibbs measure. Indeed, real coherent structures in turbulent flows are not necessarily self-avoiding: careful simulation results have shown that filaments with finite cross section can cross each other [3].

4. Persistence length

It has to be recalled that the scaling solution presented in section 2 assumes that the Reynolds number is high and that filaments are long enough. In numerical simulations, the length of a filament is usually determined with respect to a fixed threshold value of the vorticity modulus, and is somewhat arbitrary. For simplicity, we can imagine that the longest vortices should extend coherently over a length of the order of the integral scale. As the Reynolds number increases, filaments are longer, in units of a, and we predict that they should become more folded, until they reach the scaling regime (2.3). The confrontation and comparison of this scenario with simulations is difficult since numerical values of the ratio q_l/q_a are generally smaller than 10^3 or even 10^2. Indeed, the shapes of the filaments usually observed are roughly linear, rather than folded. We argue in the following that the framework of our model can explain this feature as an effect of finite Reynolds numbers.

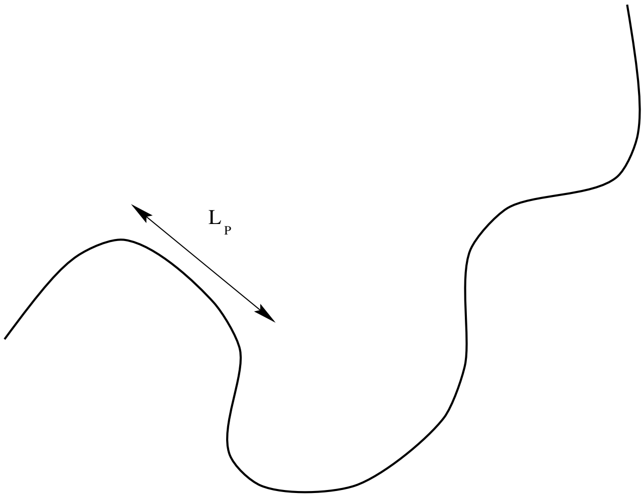

Figure 1. Schematic view of a vortex filament. The line follows a random path, but looks straight if observed at a scale smaller than a length L_p, defined as the persistence length.

Let us borrow a scaling argument often used in studies on the conformations of semi-flexible chains or charged polymers [19]. Suppose that the filaments are locally straight (or 'rigid'), such that a line fraction can be considered as linear, provided its length is smaller than a persistence length L_p. The conformation of a long filament, at scales much larger than L_p, can be figured out as a flexible chain of elementary linear units of size L_p (see figure 1). Many visualizations of filaments in simulations support this scheme; however, there, each filament contains in general a small number of persistence lengths. The scaling argument (2.3) is, in contrast, valid if the filament length L contains many persistence lengths, or $L/L_p \gg 1$. We define the persistence length as†

$$L_p = \int_0^\infty \mathrm{d}r \, R_\omega^{(f)}(r) \tag{4.1}$$

and then identify $R_\omega^{(f)}$ with $R_\omega^{(b)}$, as discussed at the end of section 2. Definition (4.1) is acceptable, provided that the major contribution to L_p comes from separation distances r smaller than those for which the scaling law (2.10) is reached: this condition is fulfilled for $\mu < 2$ (hence for a Kolmogorov spectrum), when the integral remains finite at infinite Reynolds numbers. L_p thus represents the length over which R_ω remains close to one, the scale where the filaments are fairly straight. From equation (2.8), we get

$$L_p = \frac{\pi}{2}\left(\int_0^\infty \mathrm{d}q\, q E^{(b)}(q)\right)\left(\int_0^\infty \mathrm{d}q\, q^2 E^{(b)}(q)\right)^{-1}. \tag{4.2}$$

We further investigate the behaviour of L_p with respect to the Reynolds number, supposed to be large but not infinite. We then compare our results with the simulations of Jimenez and Wray [4] and Vincent and Meneguzzi [5], where vortex filaments were observed along with a Kolmogorov energy spectrum in the inertial range, $E^{(b)}(q) = C_K \epsilon^{2/3} k^{-5/3}$ (with ϵ the energy dissipation rate and $C_K \simeq 2$ in both cases). In equation (4.2), we assume a Kolmogorov form

† An alternate definition for L_p can be introduced via the behaviour of the autocorrelation function at small r: $R_\omega(r) \simeq 1 - (r/L_p)^2$. This leads to a result of a similar order of magnitude.

in the interval $q_l < q < q_a$; in addition we set $E^{(b)} = 0$ for $q > q_a$, and for generality $E^{(b)} \sim q^{\mu_0}$ for $q < q_l$, where μ_0 is a positive constant. Equation (4.2) yields

$$L_p = L_p^\infty \frac{1 - \frac{\mu_0+1}{\mu_0+2}(q_a/q_l)^{-1/3}}{1 - \frac{\mu_0+2}{\mu_0+3}(q_a/q_l)^{-4/3}} \qquad L_p^\infty \equiv 2\pi q_a^{-1}. \tag{4.3}$$

The above result shows that $L_p q_a$ is a monotonic increasing function of q_a/q_l, and asymptotically reaches the finite value 2π. Hence, our model predicts that the persistence length in units of the dissipation scale a increases with the Reynolds number. It is consistent with the numerical observations of [4], where similar (although different from equation (4.1) in their definition) correlation lengths were introduced. Note that L_p depends only weakly on the non-universal exponent μ_0. In order to match the finite-box conditions of the simulations, we will set in the following $E^{(b)} = 0$ in the region $q < q_l$, which by continuity is equivalent to taking $\mu_0 \gg 1$. In order to make more precise comparisons, we need to rewrite L_p as a function of the integral scale l, the dissipation scale a and the microscale Reynolds number Re_λ, using the same definitions as in the numerical studies. This is required by the particular importance of scaling law prefactors, especially at intermediate Re_λ. Hence, it is necessary: (i) to express q_a (q_l) as a function of $1/a$ ($1/l$); and (ii) to relate q_a/q_l to Re_λ. In [4] and [5] the small-scale unit is the Kolmogorov scale defined as $a \equiv (\nu^3/\epsilon)^{1/4}$; in turn, the definition for the integral scale l and the Taylor microscale λ differs in the two cases from numerical factors.

The definitions of [4] applied to our truncated spectrum lead to $q_a = (3C_K/2)^{-3/4}a^{-1}$, $q_l = 3\pi/10\ l^{-1}$ and $q_a/q_l = [(3C_K/2)^{-3/4}(3/20)^{1/4}]^3 Re_\lambda^{3/2}$. Table 1 displays the results obtained after inserting these expressions into equation (4.3) for various Reynolds numbers investigated in [4]. The general evolution of L_p/a is comparable to the evolution of the correlation lengths plotted in figure 6(*b*) of [4], and the numerical values are of the same order of magnitude. Note that, in our model, the persistence length reaches a limiting value of the order of ten Kolmogorov scales at very large Re_λ:

$$L_p^\infty = 2\pi \left(\frac{3C_K}{2}\right)^{3/4} a \simeq 14.3a. \tag{4.4}$$

The asymptotic scaling proposed by Jimenez *et al* is qualitatively different, as they claimed that some of the filament correlation lengths they introduced were of the order of the Taylor microscale λ, and thus should keep increasing with Re_λ ($\lambda \sim aRe_\lambda^{1/2}$). They argued that such a scaling would be consistent with a curvature radius of order λ. Apart from the difference in the definitions used, it seems that Re_λ is not large enough to discriminate unambiguously between that estimate and our theoretical result. Indeed, our L_p happens to have numerical values close to the Taylor microscale λ in the range of the Reynolds numbers considered. Table 1 also shows that L_p is not much lower than l, so that the filament cannot be considered in the scaling regime.

The other qualitative comparison, made with the calculations of Vincent and Meneguzzi [5] at $Re_\lambda \simeq 150$, is also displayed in table 1. From their definitions, one gets $q_l = 2/5l^{-1}$ and $q_a/q_l = [(3C_K/2)^{-3/4}2^{-1/2}]^3 Re_\lambda^{3/2}$. Inserting these expressions in equation (4.3) (note that equation (4.4) still applies) gives $L_p/l = 0.21$. This shows that, in that case as well, L_p is not small compared with the integral scale or the filament average length. This conclusion is in qualitative agreement with the views of the vorticity tubes presented in [5], where vortices remain quite straight.

Table 1. Theoretical persistence length for various microscale Reynolds numbers. The parameters of the energy spectra are chosen in order to match those corresponding to the flows studied in [4] and [5].

	Re_λ	q_a/q_l	L_p/a	L_p/l
Reference [4]	35.1	4.2	6.4	0.69
	61.1	9.7	8.0	0.37
	94.1	18.6	9.1	0.22
	168.1	44.3	10.3	0.10
Reference [5]	150.0	54.8	10.6	0.21

5. Conclusion

We have presented a scaling theory describing the conformations of thin independent vortex filaments in incompressible homogeneous turbulence. A self-consistent approach predicts that the filaments have a fractal structure, provided that they are objects of very large aspect ratio, which is the case if the Reynolds number of the flow is high enough. We have derived a simple relation between the Flory exponent of a filament (or its inverse fractal dimension) and the exponent of the power-law decay with separation distance of its tangent vector autocorrelation function. Another general identity, relating the Flory exponent and the kinetic energy spectrum produced by the filament itself, was also established. We further specified the results by making the assumption that filaments are special realizations of the background flow statistics, and their vorticity correlation function is identified with that of the background. According to this hypothesis, a background flow with a $k^{-5/3}$ energy spectrum contains filaments with Flory exponent $\nu = 3/5$. Up to a scale defined as the persistence length, the filaments are locally straight. The persistence length is a more advantageous and reliable scale than the total length, which is difficult to define properly and varies among a given population of filaments. We find out that the ratio of the persistence length to the Kolmogorov scale varies slowly with the Reynolds number and is of the order of 10. This order of magnitude is in qualitative agreement with available numerical studies, and the model can explain the quite linear shape of the filaments observed at intermediate Reynolds numbers. Further comparison in the asymptotic regime is limited by the lack of numerical data at very high Reynolds numbers.

Making comparison of our results with experiments is not easy, even though the highest Reynolds numbers reached in experiments are much larger than in numerical simulations. It is conceivable that a wide family of filaments with different characteristics can actually exist at very high Re and that their main properties could even change with Re, as suggested in [7]. Furthermore, it has been pointed out that the large-scale conformations of filaments measured in experiments may not be universal, depending on the large-scale generation of turbulence [8]. For instance, direct visualizations in a von Kármán swirling flow at $Re \sim 10^5$ show spatially coherent structures that initially form straight filaments of very large aspect ratio, rather than more complex fractal objects [2]. The order of magnitude of the persistence length predicted by our model cannot explain such observations. However, these filaments have a very short life time (one turnover time) and probably belong to a different class from those tracked in the numerical studies mentioned in section 5.

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Appendix

The self-consistent approximation (2.6) is correct when the random walk obeys appropriate scaling properties. Let us write the exact relation:

$$\langle \boldsymbol{t}(\boldsymbol{0}) \cdot \boldsymbol{t}(\boldsymbol{r})\rangle = \int_0^\infty \mathrm{d}s\, P(s|r)\langle \boldsymbol{t}(0) \cdot \boldsymbol{t}(s)\rangle \tag{A.1}$$

where $P(s|r)$ is the probability density that two points of the line distant from a distance r are separated by a line segment of length s. We may apply to $P(s|r)$ the general arguments usually invoked to analyse the scaling properties of the reciprocal conditional probability $P(r|s)$ in random walk problems [14]. Together with the scaling relation (2.3), we make the assumption that $P(s|r)$ is a peaked function around the typical value $s^* \sim r^{1/\nu}$, and that it depends on s only through the ratio $s/r^{1/\nu}$. One introduces the scaling form

$$P(s|r) = \frac{1}{r^{1/\nu}}\, p\left(\frac{s}{r^{1/\nu}}\right) \tag{A.2}$$

where the factor $1/r^{1/\nu}$ ensures the normalization $\int_0^\infty p(x)\,\mathrm{d}x = 1$. In addition, from equation (2.5), it can easily be shown that, if $\nu > 1/2$, the correlation function $\langle \boldsymbol{t}(0) \cdot \boldsymbol{t}(s)\rangle$ must decay as $s^{-(2-2\nu)}$ at large s. One deduces from equations (A.1) and (A.2) that

$$\langle \boldsymbol{t}(\boldsymbol{0}) \cdot \boldsymbol{t}(\boldsymbol{r})\rangle \sim r^{-(2-2\nu)/\nu} \int_c^\infty \mathrm{d}x\, p(x) x^{-(2-2\nu)} \tag{A.3}$$

where $c < 1$ is an arbitrary cut-off. Comparing equation (A.3) with (2.4) leads to the result (2.7).

Self-consistent scheme are commonly used in diffusion problems, such as Brownian diffusion in disordered media [15]. There, the exponent ν relates the rms position of a diffusing particle with time through the relation $\langle \boldsymbol{r}^2\rangle^{1/2} \sim t^\nu$—when both thermal and disorder configuration averages are performed. Indeed, the solution represented by equation (2.7) is precisely the self-consistent result for the diffusion of a Brownian particle in a quenched random force field $\boldsymbol{F}$ with Gaussian distribution and long-range correlations, $\langle \boldsymbol{F}(\boldsymbol{x})\cdot\boldsymbol{F}(\boldsymbol{x}')\rangle \sim |\boldsymbol{x}-\boldsymbol{x}'|^{-\alpha}$. Starting from the Langevin equation, it is easy to check that that problem satisfies the second moment relation (2.4) where the curvilinear coordinate s must be replaced by the time t and the tangent vector $\boldsymbol{t}$ by the particle velocity $\mathrm{d}\boldsymbol{r}/\mathrm{d}t$ ($\boldsymbol{x}$ and $\boldsymbol{x}'$ are two positions on the particle trajectory). If $R_\omega^{(f)} = R_\omega^{(b)}$ (see the text), the background vorticity $\vec{\omega}^{(b)}$ formally corresponds to the random field $\boldsymbol{F}$. As quoted in [15], the self-consistent approximation is generally not correct for this diffusion problem, and the results actually depend on topological details of the force field. However, the self-consistent calculation of ν happens to be *exact* for divergence-less fields, $\nabla \cdot \boldsymbol{F} = 0$ [20].

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